SOLVING OF TWO-DIMENSIONAL STABLE THERMAL CONDUCTION PROBLEM WITHOUT AN INTERNAL HEAT SOURCE USING FINITE DIFFERENCE METHOD

GIẢI BÀI TOÁN DẪN NHIỆT ỔN ĐỊNH HAI CHIỀU KHÔNG CÓ NGUỒN NHIỆT BÊN TRONG BẰNG PHƯƠNG PHÁP SAI PHÂN HỮU HẠN

MS. Thi - Anh - Dao Nguyen
Faculty of Fundamental Science - Mien Tay Construction University
Email: nguyenthianhdao@mtu.edu.vn
Phone: 0916 804 508

Ngày nhận bài: 28/02/2023
Ngày gửi phản biện: 13/03/2023
Ngày chấp nhận đăng: 20/03/2023

Tóm tắt:

Bài viết trình bày các bước thiết lập công thức tính nhiệt độ tại một trong các điểm nút bất kỳ trong mạng các đường vuông góc hai chiều x, y. Xét trường hợp dẫn nhiệt ổn định không có nguồn nhiệt bên trong, trường nhiệt độ được mô tả bởi toán tử Laplace trong hệ tọa độ Descartes hai chiều. Phương pháp sai phân hữu hạn (FDM) được áp dụng để tính giá trị nhiệt độ của bốn nút trong lưới x, y trên tấm hai chiều. Các phương trình nút được viết và phương pháp Gauss cũng được sử dụng để giải các hệ phương trình này và kết quả của chúng là nhiệt độ tại các nút khác nhau.

Từ khóa: *Dẫn nhiệt ổn định hai chiều, Phương pháp sai phân hữu hạn, Phương pháp Gauss.*

Abstract:

This article presents the steps to establish the formula for calculating temperatures at any of the nodal points within in the network of two-dimensional perpendicular lines x, y. Consider in the case of stable heat conduction without an internal heat source, the temperature field is described by the Laplace operator in the two-dimensional Cartesian coordinate system. Finite difference method (FDM) is adopted to calculate the temperature value of four nodes in the x, y grid on a two-dimensional plate. The nodal equations are written, and Gauss method is also used to solve these system of equations and their results are the temperatures at the various nodes.

Keywords: *Two-dimensional steady-state heat Conduction, Finite Difference Method, Gauss Method.*

1. Issue

In many fields of engineering heat transfer by conduction, convection, and radiation are considered as crucial processes.

Heat transfer is a thermal movement that facilitates the exchange of energy between materials (solid/liquid/gas) as a result of a temperature difference [1]. Heat conduction, also called diffusion, is the method of transferring energy from higher energy particles to lower energy particles by collision of particles or lattice vibrations [1].

This is a physical process in which heat propagates from a temperature source along a section of area or volume, producing thermal gradients. This phenomenon can be classified as a steady state when the heat transmitted in a system is constant and only the temperature varies at each point in the system.

This article is targeted solely on only the heat transfer by conduction in steady state condition through a two-dimensional rectangular plate. With the application of the Finite Difference Method (FDM) [1], it is possible to solve heat transfer problems by conduction numerically in a relatively fast. This is a numerical method that can be used to solve differential equations by discretizing the continuous physical domain in a finite discrete mesh [2]. Besides, the Gauss Method is also applied to solve the temperature calculation of discrete points in this problem [3].

2. Content

2.1. Problem for two-dimensional steady state heat conduction

Depending on the magnitude of the conduction in each direction and the determination of the problem solution, heat transmission can be classified into one-direction, two-direction and three-direction. As mentioned in the introduction to the article, temperature distribution problems inside a plate without heat generation with the two-dimensional (2D) mesh conduction mechanisms are addressed, considering in a steady state. The general form of heat equation can be applied to two dimensional thermal problems with boundary conditions of type one.

In this section we consider the numerical formulation and solution of two-dimensional steady heat conduction in rectangular coordinates using the finite difference method.

A rectangular plate of x mm X y mm. The grid size assumed is in both directions. This plate is insulated everywhere. Besides, the temperature can be set at a prescribed level at edges, it indicates that heat transfer is limited to the x and y dimensions. The objective is to analyze the interior points temperatures (T_1, T_2, T_3 & T_4). Figure 1 shows an element on the face of a thin rectangular plate of thickness.

ISSN 2525-2615

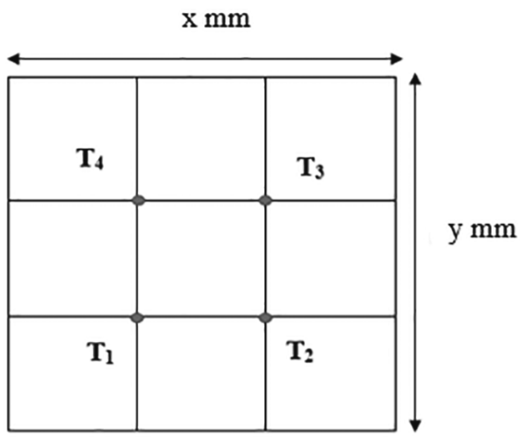

Fig.1. *Rectangular plate*

The Laplace equations are typically used to characterize steady-state, boundary value problems [2]. We will illustrate a simple case - The Laplace equation in two-dimensional steady state heat conduction is given by

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0 \qquad (1)$$

The application of the Finite Difference Method (FDM) [4], two dimensional thermal problems with boundary conditions of type one can be solved.

Finite-difference representations based on treating the plate as a grid of discrete points are substituted for the partial derivatives in Eq. (1).

Now consider a rectangular region for t (x, y) which forms the closed boundary in Fig. 2.

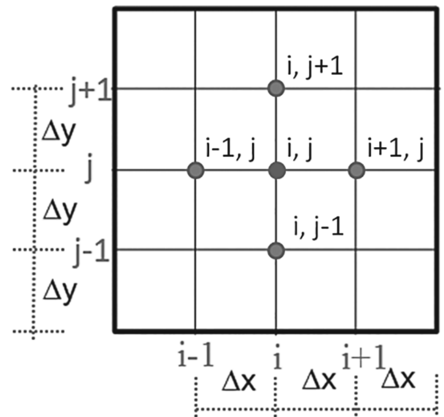

Fig.2. *Two-dimensional numerical analysis of heat conduction*

With substitutions of the first order derivative by the finite difference method, we obtain:

$$\frac{\partial t}{\partial x} \approx \frac{\Delta t}{\Delta x} = \frac{t_{i,j} - t_{i-1,j}}{\Delta x} \qquad (2)$$

Similarly, we can write

$$\frac{\partial^2 t}{\partial x^2} \approx \frac{\Delta(\Delta t)}{\Delta x \Delta x} = \frac{t_{i+1,j} + t_{i-1,j} - 2t_{i,j}}{(\Delta x)^2} \qquad (3)$$

$$\frac{\partial^2 t}{\partial y^2} \approx \frac{\Delta(\Delta t)}{\Delta y \Delta y} = \frac{t_{i,j-1} + t_{i,j+1} - 2t_{i,j}}{(\Delta y)^2} \qquad (4)$$

To find (1), we can substitute Eq. (3) and Eq. (4) into Eq. (1) results in

$$\frac{t_{i+1,j} + t_{i-1,j} - 2t_{i,j}}{(\Delta x)^2} + \frac{t_{i,j-1} + t_{i,j+1} - 2t_{i,j}}{(\Delta y)^2} = 0 \qquad (5)$$

If we divide the region into square region of each, then the Laplace equation can be written as

$$t_{i,j} = \frac{t_{i+1,j} + t_{i-1,j} + t_{i,j-1} + t_{i,j+1}}{4} \qquad (6)$$

This relationship, which holds for all

interior points on the plate, is referred to as the Laplacian difference equation [1].

In addition, boundary conditions along the edges of the plate must be specifi ed to obtain a unique solution. The simplest case is where the temperature at the boundary is set at a fixed value [1]. This is called a Dirichlet boundary condition.

Note that for the case where there are sources or sinks of heat within the two-dimensional domain, the equation can be represented as

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = f(x, y) \qquad (7)$$

where f (x, y) is a function describing the sources or sinks of heat. Equation (7) is referred to as the Poisson equation [1].

2.2. Example

We consider an example with a concrete beam having the cross section shown is fig 3. In fig. 3, temperature conditions along the edges of the plate are held at constant levels. The objective is to analyze the interior points temperatures (T_1, T_2, T_3 & T_4).

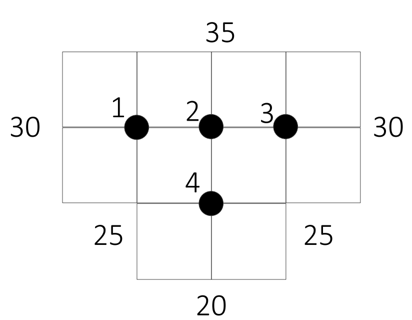

Fig. 3. *Temperature distribution for a concrete beam subject to fixed boundary conditions* [5].

Applying Eq. (6) at node (i, j) gives

$$t_{i,j} = \frac{t_{i+1,j} + t_{i-1,j} + t_{i,j-1} + t_{i,j+1}}{4}$$

For nodes 1, 2, 3, 4, Eq. (6) can be used to compute

$4t_1 = 30 + 35 + t_2 + 25$

$4t_2 = t_1 + 35 + t_3 + t_4$

$4t_3 = t_2 + 35 + 30 + 25$

$4t_4 = t_2 + 25 + 20 + 25$

We have a system with four equations with four unknowns:

$$\begin{cases} 4t_1 - t_2 = 90 \\ -t_1 + 4t_2 - t_3 - t_4 = 35 \\ -t_2 + 4t_3 = 90 \\ -t_2 + 4t_4 = 70 \end{cases}$$

Or

$$\begin{cases} 4t_1 - t_2 + 0t_3 + 0t_4 = 90 \\ -t_1 + 4t_2 - t_3 - t_4 = 35 \\ 0.t_1 - t_2 + 4t_3 + 0t_4 = 90 \\ 0t_1 - t_2 + 0t_3 + 4t_4 = 70 \end{cases} \qquad (8)$$

In order to make simple, this linear equation system can be solved by the Gauss method.

2.3. The Gauss Method

One of the most commonly employed approach in two-dimensional steady state heat conduction Problem is Gauss method. This method used to solve the linear equation systems which include elementary transformation consists in bringing the initial equa-tion system in equivalent form [6].

On the first, the transformation of the initial system is made in a triangle system form. We can eliminate the

ISSN 2525-2615

unknown from equations with three elementary operations as

• Swap the positions of the two equations each other.

• Product non-zero constant with one equation.

• The difference between two equations and replace the second equation with the result of this operation.

As a next step, we solve triangle matrix system consists in determination of the unknowns and substitution them in equations of the system in inverse order. This phase is inverse substitution phase, or it means the backtracking method. The Gauss method can be generalized and applied to the solution of m systems of n linear equations with n unknown.

The Gauss method can be detailed to solve Eq (8), where we consider the matrix contains four rows and four columns.

Eq. (8) becomes

$$\begin{cases} -t_1 + 4t_2 - t_3 - t_4 = 35 \\ 0.t_1 - t_2 + 4t_3 + 0t_4 = 90 \\ 0t_1 - t_2 + 0t_3 + 4t_4 = 70 \\ 4t_1 - t_2 + 0t_3 + 0t_4 = 90 \end{cases} \quad (9)$$

Extended matrix form is written:

$$\underline{A} = (35\ 90\ 70\ 90\ ) \quad (10)$$

We use elementary transformations (line-by-line) to bring the matrix A to the ladder matrix

$$\underline{A}\ d_4 \rightarrow d_4 + 4d_1 \rightarrow\ (35\ 90\ 70\ 230\ )\ d_3$$
$$\rightarrow d_3 - d_2\ d_4$$
$$\rightarrow d_4 + 15d_2$$
$$\rightarrow\ (35\ 90\ \ -20\ 1580\ )$$

$d_4 \rightarrow d_4 + 14d_3 \rightarrow\ (35\ 90\ \ -$
$20\ 1300\ )$.

We can rewrite Eq. (9)

$$\begin{cases} t_1 + 4t_2 - t_3 - t_4 = 35 \\ 0t_1 - t_2 + 4t_3 + 0t_4 = 90 \\ 0.t_1 + 0t_2 - 4t_3 + 4t_4 = -20 \\ 0t_1 + 0t_2 + 0t_3 + 52t_4 = 1300 \end{cases} \quad (11)$$

From the obtained results, we can easily find the solution of Eq. (9)

$$\begin{cases} t_1 = 30 \\ t_2 = 30 \\ t_3 = 30 \\ t_4 = 25 \end{cases} \quad (12)$$

So, the interior points temperatures (T_1, T_2, T_3& T_4) were calculated through above calculation and analysis.

Consequently, through the calculation and analysis, we can apply the mathematical technique performed for the prediction of the temperature distribution at the inside points of the rectangular plate.

3. Conclusions

In studies of heat conduction issues, the two-dimensional thermal problems have many applications in many fields of science and engineering like the construction engineering, bioengineering, power engineering, chemical engineering, nuclear physics, and so on. This article contains the description of the heat equation is given by the Laplace equation in two-dimensional steady state condition.

The content compiled help readers get used to problem for heat networks. Also, it is one of the most crucial lectures the module Heat Transfer Engineering for the bilingual training program and international student.

The article is mainly based on the references No.1. From these, we have selected contents close to Chapter 1. Heat conduction: as well as the existing lecture in Vietnamese. Although there are a lot of great efforts, some errors are unavoidable. I look forward to receiving valuable suggestions from colleagues and readers to make it better.

References

[1] G. F. Nellis and S. A. Klein, *Introduction to Engineering Heat Transfer*. Madison: University of Wisconsin, 2020;

[2] C. Balaji, Balaji Srinivasan and Sateesh Gedupudi, *Heat Transfer Engineering: Fundamentals and Techniques*. Indian Institute of Technology Madras, Chennai, Tamil Nadu, India, 2020;

[3] Trịnh Văn Quang, *Kỹ thuật nhiệt*, NXB Khoa học và Kỹ thuật, 2007;

[4] Võ Chí Chính, *Kỹ thuật nhiệt*, NXB Khoa học và Kỹ thuật, 2006;

[5] Bộ môn Lý-Hóa, *Bài giảng Nhiệt kỹ thuật*, Vĩnh Long: Trường ĐHXD Miền Tây, 2021;

[6] Bộ môn Toán-tin, *Bài giảng Toán 1*, Vĩnh Long: Trường ĐHXD Miền Tây, 2020.

ISSN 2525-2615